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INTERPOLATING SPLINES AS LIMITS OF POLYNOMIALS.(U)
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INTERPOLATING SPLINES AS LIMITS OF POLYNOMIALS

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July 1981

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ABSTRACT

Let the points

(1) (x_i, y_i), $(i = 1,\ldots,k;\ k \geq 2)$, $a \leq x_1 < x_2 < \cdots < x_k \leq b$,

$$I = [a,b], \qquad (-\infty < a < b < \infty)$$

be prescribed. Furthermore, let m and n be integers such that

$$1 \leq n < k \leq m ,$$

and define the polynomial class

$$\Pi_m = \{P(x);\ P(x) \in \pi_m,\ P(x_i) = y_i,\ \ (i = 1,\ldots,k)\} .$$

Within Π_m we determine $P_m(x)$ as the solution of the extremum problem

(2) $$\int_I (P^{(n)}(x))^2 dx = \text{minimum} \quad \text{for} \quad P(x) \in M_m .$$

Finally, let $S(x) = S_{2m-1}(x)$ be the natural spline interpolant of degree $2n - 1$ of the k points (1). Our main result is

Theorem 1. 1. There is a unique polynomial $P_m(x)$ which is the solution of the minimum problem (2).

2. We have

$$\lim_{m\to\infty} P_m(x) = S(x) \quad \text{uniformly in} \quad x \in I .$$

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SIGNIFICANCE AND EXPLANATION

In the finite interval $[a,b]$ we have prescribed abscissae $< x_2 < \cdots < x_k$ and corresponding ordinates $y_1, y_2, \ldots, y_k$. Let $S(x) = S_{2k-1}(x)$ be the natural spline of degree $2n - 1$ that interpolates those k points. This requires that $1 \leq n < k$. Furthermore, let m be an integer such that $m \geq k$, and let $P_m(x)$ be the polynomial of degree at most m that interpolates the k points, and such as to minimize the integral

$$\int_a^b (P^{(n)}(x))^2 dx ,$$

within the entire class of polynomials of degree m that interpolate the k points. It is shown that as $m \to \infty$ the polynomial $P_m(x)$ converges to the spline $S(x)$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

INTERPOLATING SPLINES AS LIMITS OF POLYNOMIALS

I. J. Schoenberg

For Alexander Ostrowski on his 90th birthday on September 25, 1983, from one of his grateful students.

1. Introduction. Let the points

$$(x_i,y_i),\ (i = 1,\dots,k;\ k \geq 2),\ a \leq x_1 < x_2 < \cdots < x_k \leq b\ , \tag{1.1}$$

$$I = [a,b], \quad (-\infty < a < b < \infty)\ ,$$

be prescribed. The basic interpolant is the Lagrange interpolating polynomial. If additional consecutive derivatives at the points (1.1) are available, we can construct the Hermite interpolation polynomial. In the absence of such additional data, we propose here the following construction:

Let m and n be integers such that

$$1 \leq n < k \leq m\ , \tag{1.2}$$

and let us consider the polynomial class

$$\Pi_m = \{P(x);\ P(x) \in \pi_m,\ P(x_i) = y_i,\ (i = 1,\dots,k)\}\ . \tag{1.3}$$

Within this class we determine the polynomial $P_m(x)$ which is "most nearly a polynomial of degree at most $n - 1$ in the interval I." We interpret this requirement to mean the $P_m(x)$ is the solution of the extremum problem

$$\int_I (P^{(n)}(x))^2 dx = \text{minimum} \quad \text{for} \quad P(x) \in \Pi_m\ . \tag{1.4}$$

Equivalently: Writing

$$M_{n,m} = \inf_{P\in\Pi_m} \int_I (P^{(n)}(x))^2 dx\ , \tag{1.5}$$

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the polynomial $P_m(x)$ *is uniquely defined by*

(1.6) $$\int_I (P_m^{(n)}(x))^2 dx = M_{n,m}, \quad P_m(x) \in \Pi_m .$$

Of course, the existence and uniqueness of $P_m(x)$ is yet to be established.

Our main subject is the behavior of $P_m(x)$ as $m \to \infty$. The statement of our result (Theorem 1 below) requires some known properties of *natural spline interpolation*. We describe its definition and the three properties that we need.

I. *Let* n *be an integer such that*

(1.7) $$1 \leq n < k ,$$

and let $S(x) = S_{2n-1}(x)$ *denote a function satisfying the following four conditions:*

1° $S(x) \in C^{2n-2}(\mathbb{R})$,

2° $S(x) \in \pi_{2n-1}$ *in each interval* (x_j, x_{j+1}), $(j = 1,\ldots,k-1)$,

3° $S(x) \in \pi_{n-1}$ *in* $(-\infty, x_1)$, *and* $S(x) \in \pi_{n-1}$ *in* $(x_k, +\infty)$,

4° $S(x_i) = y_i$, $(i = 1,\ldots,k)$.

Then $S(x)$ *is uniquely defined by the conditions* 1° to 4°.

The function $S(x)$ is called the *natural spline interpolant of the points* (1.1) *of degree* $2n-1$.

II. *If* $f(x) \in C^{n-1}(I)$ *is such that*

(1.8) $$f(x_i) = y_i, \quad (i = 1,\ldots,k) ,$$

(1.9) $$f^{(n-1)}(x) \text{ is absolutely continuous}, \quad f^{(n)}(x) \in L_2(I) ,$$

then

(1.10) $$\int_I (f^{(n)}(x))^2 dx \geq \int_I (S^{(n)}(x))^2 dx ,$$

with the equality sign only if $f(x) = S(x)$ *in* I.

III. If $f(x) \in C^{n-1}(I)$ satisfies (1.8) and (1.9), then

$$\int_I (f^{(n)})^2 dx - \int_I (S^{(n)})^2 dx = \int_I (f^{(n)} - S^{(n)})^2 dx \quad (1.11)$$

(See for instance [1, 110-116]).

Our main result is

Theorem 1. 1. There is a unique polynomial $P_m(x)$ satisfying (1.6), where $M_{n,m}$ is defined by (1.5).

2. We have

$$\lim_{m\to\infty} P_m(x) = S(x) \text{ uniformly in } x \in I . \quad (1.12)$$

In view of the extremum property II of $S(x)$, the limit relation (1.12) may not seem surprising. Even so it is no immediate consequence and our proof of Theorem 1 occupies the remaining three sections of this note.

2. The existence and uniqueness of $P_m(x)$. 1. Without loss of generality we may restrict the search for $P_m(x)$ to the subclass $\Pi_m^* \subset \Pi_m$ of polynomials $P(x) = \sum_0^m a_r x^r/r!$ satisfying

$$\int_I (P^{(n)}(x))^2 dx \leq M_{n,m} + 1 , \quad (2.1)$$

where

$$P^{(n)}(x) = \sum_{r=n}^{m} a_r x^{r-n}/(r-n)! . \quad (2.2)$$

Let $X_i(x)$, $(i = 0,1,\ldots)$ be the orthonormal polynomials for the interval I, and let

(2.3) $$P^{(n)}(x) = \sum_{0}^{m-n} c_i X_i(x) .$$

From (2.1) and Parseval's theorem we conclude that $\sum_{0}^{m-n} c_i^2 \leqq M_{n,m} + 1 = K^2$, and hence that $|c_i| \leqq K$, $(i = 0,\ldots,m - n)$. From (2.3) it follows that the coefficients

(2.4) $$a_n, a_{n+1}, \ldots, a_m \text{ are bounded .}$$

Because $n < k$ we have that $P(x_i) = y_i$ for $i = 1,\ldots,n$. Solving this system for the unknowns $a_0,\ldots,a_{n-1}$, in terms of the coefficients (2.4), we conclude that for an appropriate constant M we have

$$|a_i| \leqq M, \quad (i = 0,\ldots,m) .$$

Now familiar compactness arguments will insure the existence of $P_m(x)$ satisfying (1.6) and (1.5).

2. Let $p_0(x)$ and $p_1(x)$ be two polynomials in Π_m such that

(2.5) $$\int_I (p_0^{(n)})^2 dx = \int_I (p_1^{(n)})^2 dx = M_{n,m} .$$

Evidently also

(2.6) $$p_t(x) = (1 - t)p_0(x) + tp_1(x) \in \Pi_m, \quad (0 \leqq t \leqq 1)$$

and

(2.7) $$\varphi(t) = \int_I ((1 - t)p_0^{(n)}(x) + tp_1^{(n)}(x))^2 dx - M_{n,m}$$

is a quadratic polynomial in t which is seen to satisfy the equations

(2.8) $$\varphi(0) = 0, \quad \varphi(1) = 0 .$$

Moreover, by (2.7)

(2.9) $$\varphi(t) = t^2 \int_I (p_1^{(n)} - p_0^{(n)})^2 dt + At + B .$$

Let us show that the inequality

$$\int_I (p_1^{(n)} - p_0^{(n)})^2 dt > 0$$

<u>is impossible</u>. Indeed, it would imply by (2.8) and (2.9), that $\varphi(t_0) < 0$ for some t_0 with $0 < t_0 < 1$. But then, by (2.7), we would have $\int_I (p_{t_0}^{(n)})^2 dx < M_{n,m}$, contradicting the definition of $M_{n,m}$ as the minimum. We must therefore have

$$\int_I (p_1^{(n)} - p_0^{(n)})^2 dx = 0, \quad \text{hence} \quad p_1^{(n)}(x) = p_0^{(n)}(x) .$$

But then $p_1(x) = p_0(x) + R(x)$, where $R(x) \in \pi_{n-1}$. Since $R(x_i) = 0$ for $i = 1,\ldots,k$, and k exceeds the degree $n - 1$ of $R(x)$, we conclude that $R(x) = 0$ and therefore $p_0(x) = p_1(x)$.

3. <u>Proof that</u> $P_m^{(n)}(x) \to S^{(n)}(x)$, <u>as</u> $m \to \infty$, <u>in the</u> L_2<u>-norm</u>. <u>Let us show that</u>

$$\lim_{m\to\infty} \int_I |P_m^{(n)}(x) - S^{(n)}(x)|^2 dx = 0 . \tag{3.1}$$

From the Property III, in particular (1.11) applied to $f(x) = P_m(x)$, we obtain

$$\int_I (P_m^{(n)}(x))^2 dx - \int_I (S^{(n)}(x))^2 dx = \int_I (P_m^{(n)}(x) - S^{(n)}(x))^2 dx . \tag{3.2}$$

The definition (1.5) of

$$M_{n,m} = \int_I (P_m^{(n)}(x))^2 dx \tag{3.3}$$

as a minimum, and (3.2), show that

$$\int_I (P_m^{(n)}(x) - S^{(n)}(x))^2 dx = \min_{P\in\Pi_m} \int_I (P^{(n)}(x) - S^{(n)}(x))^2 dx . \tag{3.4}$$

Clearly, the class Π_m expands on increasing m; this shows that $M_{n,m}$ is non-increasing for increasing m, and by (3.2) also the right side of (3.2) form a non-increasing sequence. This insures the existence of the non-negative limit

(3.5) $$\lim_{m\to\infty} \int_I (P_m^{(n)} - S^{(n)})^2 dx = L .$$

A proof of (3.1) is now equivalent to showing that

(3.6) $$L = 0 .$$

This requires two lemmas from Approximation Theory, the first of which is well known as an easy consequence of Weierstrass' theorem.

Lemma 1. Given $\varepsilon > 0$ we can find a polynomial $P_*(x)$ such that

(3.7) $$|S(x) - P_*(x)| < \varepsilon \text{ and } |S^{(n)}(x) - P_*^{(n)}(x)| < \varepsilon \text{ in } I .$$

Indeed, if in the relation

$$S(x) = \sum_0^{n-1} S^{(r)}(a)(x-a)^r/r! + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}S^{(n)}(t)dt .$$

We approximate to $S^{(n)}(t)$ closely by a polynomial $p(t)$, then the polynomial

$$P(x) = \sum_0^{n-1} S^{(r)}(a)(x-a)^r/r! + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}p(t)dt$$

will also approximate closely to $S(x)$. Since $P^{(n)}(t) = p(t)$, the lemma follows.

Lemma 2. Given $\delta > 0$, we can find an $m \geq k$, and a $P(x) \in \Pi_m$, such that

(3.8) $$|S^{(n)}(x) - P^{(n)}(x)| < \delta \text{ for } x \in I .$$

Notice that $P(x) \in \Pi_m$ requires that $P(x_i) = y_i$. This we derive from Lemma 1 by Lagrange interpolation as follows. Let $P_*(x)$ be the polynomial of Lemma 1 satisfying (3.7), and let

$$m = \max(k, \text{ degree of } P_*) , \tag{3.9}$$

hence $P_*(x) \in \pi_m$ and $m \geq k$. From $S(x_i) = y_i$ and the first inequality (3.7) we have

$$|P_*(x_i) - y_i| < \varepsilon, \quad (i = 1,\dots,k) . \tag{3.10}$$

Let $Q(x) \in \pi_{k-1}$ be such that

$$Q(x_i) = P_*(x_i) - y_i, \quad (i = 1,\dots,k) . \tag{3.11}$$

Finally, we define

$$P(x) = P_*(x) - Q(x) . \tag{3.12}$$

Notice that $S^{(n)}(x) - P^{(n)}(x) = S^{(n)}(x) - P_*^{(n)}(x) + Q^{(n)}(x)$ and therefore

$$|S^{(n)}(x) - P^{(n)}(x)| \leq |S^{(n)}(x) - P_*^{(n)}(x)| + |Q^{(n)}(x)| \quad (x \in I) . \tag{3.13}$$

At this point observe that by (3.11) we have by Lagrange's formula

$$Q(x) = \sum_1^k \ell_i(x)(P_*(x_i) - y_i)$$

and therefore also the inequality

$$\|Q^{(n)}(x)\|_\infty \leq K_n \max_i |P_*(x_i) - y_i| \quad \text{where} \quad K_n = \max_x \sum_i |\ell_i^{(n)}(x)| . \tag{3.14}$$

By (3.7), (3.13), and (3.14), we conclude that

$$|S^{(n)}(x) - P^{(n)}(x)| \leq (1 + K_n)\varepsilon . \tag{3.15}$$

Clearly, $P(x)$ satisfies (3.8) if we choose $\varepsilon = \delta/(1 + K_n)$.

The $P(x)$ defined by (3.12) satisfies all conditions required by Lemma 2: $P(x) \in \pi_m$ by (3.12). Also $P(x) \in \Pi_m$, because by (3.11) $P(x_i) = P_*(x_i) - Q(x_i) = P_*(x_i) - P_*(x_i) + y_i = y_i$. This completes a proof of Lemma 2.

A proof of (3.6) follows at once, because by (3.4) we have

$$0 \leq L \leq \int_I (P_m^{(n)} - S^{(n)})^2 dx \leq \int_I (P^{(n)} - S^{(n)})^2 dx \leq \delta^2 (b - a) ,$$

where δ is arbitrarily small.

4. A proof of the limit relation (1.12). Newton's formula with divided differences

$$S(x) = S(x_1) + (x - x_1)S(x_1,x_2) + \cdots + (x - x_1) \cdots (x - x_{n-1})S(x_1,\ldots,x_n)$$
$$+ (x - x_1) \cdots (x - x_n)S(x_1,\ldots,x_n,x)$$

shows the following: If $Q_0(x) \in \pi_{n-1}$ denotes the Lagrange interpolation of $S(x)$ at the points $x_1,\ldots,x_n$, then

(4.1) $$S(x) = Q_0(x) + (x - x_1) \cdots (x - x_n)S(x_1,\ldots,x_n,x) .$$

This is possible because of (1.2), hence $n < k$.

Now we use the expression of divided differences in terms of B-splines: If

(4.2) $$M(t) = M(t;x_1,x_2,\ldots,x_n,x) \qquad (x \in I)$$

is the B-spline of degree $n - 1$ based on the $n + 1$ knots $x_1,\ldots,x_n,x$, then

(4.3) $$S(x_1,\ldots,x_n,x) = \frac{1}{n!} \int_I M(t)S^{(n)}(t)dt .$$

(See e.g. [1, p. 112]. In that paper B-splines are still called fundamental splines.) Applying (4.1), (4.2), and (4.3) to $S(x)$, as well as $P_m(x)$, and subtracting one equation from the other, we obtain that

(4.4) $$P_m(x) - S(x) = \frac{1}{n!} \prod_1^n (x - x_j) \cdot \int_I M(t)(P_m^{(n)}(t) - S^{(n)}(t))dt .$$

Applying the Schwarz inequality we obtain

$$(4.5) \qquad |P_n(x) - S(x)|^2 \leq (n!)^{-2} \prod_1^n (x - x_j)^2 \int_I M(t;x_1,\ldots,x_n,x)^2 dt \cdot$$

$$\cdot \int_I (P_m^{(n)}(t) - S^{(n)}(t))^2 dt .$$

Since

$$(n!)^{-2} \prod_1^n (x - x_j)^2 \int_I M(t;x_1,\ldots,x_n,x)^2 dt$$

is certainly a continuous function of the variable $x \in I$, it is also bounded. Therefore there is a constant H^2 such that (4.5) gives the estimate

$$|P_m(x) - S(x)|^2 \leq H^2 \int_I (P_m^{(n)}(t) - S^{(n)}(t))^2 dt \quad \text{for} \quad x \in I .$$

Now the relation (3.1) completes our proof of (1.12).

5. Numerical examples. The explicit evaluation of the polynomial $P_m(x)$ is an elementary problem of linear algebra in $m + 1$ unknowns. This is the reason why Theorem 1 is so welcome: It replaces for large m, the construction of $P_m(x)$ by the much simpler construction of $S(x)$. We may say that Theorem 1 adds to the interest that we attribute to the natural spline interpolant $S(x) = S_{2n-1}(x)$.

The unicity of $P_m(x)$ in Theorem 1 clearly implies that if the data (1.1) are symmetric about the origin, i.e. $b = -a$, $x_i = -x_{k-i+1}$, then $P_m(x)$ must be an even polynomial, hence $P_{2r+1}(x) = P_{2r}(x)$.

For our examples we choose the simplest such symmetric case, namely

$$k = 3,\ (a,b) = (-1,1),\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ y_1 = 1,\ y_2 = 0,\ y_3 = 1 .$$

Selecting $n = 1$, and $m = 3, 4, 5, 6,$ and 7, we find by elementary calculations that

$$P_3(x) = x^2 ,$$

$$P_4(x) = P_5(x) = (18/11)x^2 - (7/11)x^4 ,$$

$$P_6(x) = P_7(x) = (25/11)x^2 - (25/11)x^4 + x^6 ,$$

while the natural spline interpolant is the linear spline $S(x) = |x|$, $-1 \leq x \leq 1$. The sequence of values

$$P_3(1/2) = .25,\ P_4(1/2) = P_5(1/2) = .37,\ P_6(1/2) = P_7(1/2) = .44 ,$$

which converge to $S(1/2) = .5$, illustrates Theorem 1.

REFERENCES

1. I. J. Schoenberg, On interpolation by spline functions and its minimal properties, ISNM, vol. 5 (1964), On Approximation Theory.

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20. ABSTRACT - Cont'd.

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